Visible actions on flag varieties and a generalization of the Cartan decomposition

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Theory of visible actions on complex manifolds, introduced by T. Kobayashi

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1. $\exists S \subset X$ *s.t.* $X' := G \cdot S$ *is open in* X.
2. $\exists \sigma : X' \to X'$ *an anti-holo. diffeo. s.t.* $\sigma|_S = \mathrm{id}_S$ *and* $\sigma(G \cdot x) \subset G \cdot x \ \forall x \in X'$.

Aim: Uniform treatment of multiplicity-free (M.F.) representations of Lie groups

Definition (M.F. representation)

Let $\mathbf{G}$ be a loc. cpt gp, $\mathbf{V}$ a unitary rep'n of $\mathbf{G}$. We say $\mathbf{V}$ is M.F. if $\mathbf{End}_{G}(\mathbf{V})$ is commutative.

Definition (M.F. representation)

Let G be a loc. cpt gp, V a unitary rep'n of G.
We say V is M.F. if $\mathrm{End}_G(V)$ is commutative.

If $\dim_{\mathbb{C}}(V)$ is finite,

$$V \text{ is M.F.} \Leftrightarrow V = \sum_{\pi \in \widehat{G}} m_\pi V_\pi, \ \ m_\pi \leq 1 \ (\forall \pi).$$

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Visible action $\rightsquigarrow$ M.F. rep'n

Theorem (Kobayashi ('13))

G: Lie gp,

$\mathcal{W} \to X$: holo. Hermitian G-vector bd'l,

V: unitary rep'n of G.

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"Propagation theorem of M.F. property"

In the statement of the propagation theorem, we do not need to assume that

- G is compact, reductive,...
- V is of finite dim'l, discretely decomposable,...or
- X is cpt.

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   K: *max. cpt subgp.*

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- The G-action $G \curvearrowright G_{\mathbb{C}}/K_{\mathbb{C}}$ is strongly visible (Kobayashi ('05)).
- There exists a G-embedding $L^2(G/K) \hookrightarrow \mathcal{O}(U)$ (Krötz–Stanton ('05)).
  Here $U \subset G_{\mathbb{C}}/K_{\mathbb{C}}$ is the complex crown of G/K (Akhiezer–Gindikin ('90)).

Example (M.F. representation)

2. G: *simple gp of Hermitian type,*
   H: *symmetric subgroup* ($H = G^{\tau}$, $\tau^2 = \mathrm{id}_G$),
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- π can be realized in $\mathcal{O}(G/K, \mathcal{L})$.
- The visibility of $H \curvearrowright G/K$ (Kobayashi ('07)) follows from the Cartan decomposition in the symmetric setting $G = HAK$.
  (Flensted-Jensen ('78), Hoogenboom ('83), T. Matsuki ('95, '97).)

The Cartan decomposition $G = KAK$ was introduced by É. Cartan ('27).

Example

$G = GL(n, \mathbb{R})$, $K = O(n) = G^{\tau}$ $(\tau(g) = {}^{T}g^{-1})$,
$A = diag(n, \mathbb{R})_{>0}$.

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$\forall g \in G$, $\exists k \in K$, $\exists x \in$ *Symm* $(n, \mathbb{R})_{>0}$ s.t. $g = kx$.

$\exists h \in K$, $\exists a \in A$ s.t. $x = hah^{-1}$.

Hence $g = khah^{-1} \in KAK$.

This means $G = KAK$.

For any reductive group G and its symmetric cpt subgp $K = G^{\tau}$, we have $G = KAK$ with A an abelian subgp.

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3. *G: simple gp of Herm. type,*
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- π can be embedded into $\mathcal{O}(G/K, \mathcal{L})$.
- The visibility of $N \curvearrowright G/K$ (Kobayashi'05) follows from the Iwasawa decomposition $G = NAK$.

Problem

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Classify visible actions.

Classification of visible action

- Let (G, K) be a Herm. sym. pair, (G, H) a sym. pair. Then $H \curvearrowright G/K$ is strongly visible (Kobayashi ('07)).

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- Let $(G_{\mathbb{C}}, V)$ be a linear M.F. space of a cpx. reductive alg. gp, G its cpt real form. Then $G \curvearrowright V$ is strongly visible (A. Sasaki ('09,'11)).

Classification of visible action

- Let $G_{\mathbb{C}}/H_{\mathbb{C}}$ be one of the following spherical varieties.

$$SL(2n+1,\mathbb{C})/Sp(n,\mathbb{C}),$$
$$SO(2n+1,\mathbb{C})/GL(n,\mathbb{C}),$$
$$Sp(n,\mathbb{C})/(\mathbb{C}^* \times Sp(n-1,\mathbb{C})),$$
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Then the action $G \curvearrowright G_{\mathbb{C}}/H_{\mathbb{C}}$ of a cpt real form G is strongly visible. (Sasaki ('11-'13)).

$G_{\mathbb{C}}$: cpx reductive alg. gp, B: Borel subgp
($G_{\mathbb{C}} = GL(n, \mathbb{C})$, $B = \{$upper triangular matrices$\}$),
X: cpx alg. variety with $G_{\mathbb{C}}$-action.

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$G_{\mathbb{C}} \curvearrowright X$ is spherical if B has an open orbit on X.
Rem: Any cpx symmetric space
(e.g. $GL(n, \mathbb{C})/(GL(p, \mathbb{C}) \times GL(q, \mathbb{C}))$) is spherical.

Spherical variety

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Fact (c.f. J. Wolf's book ('07))

If (G, H) is a reductive Gel'fand pair, $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical.

Exa: $(G, H) = (GL(n, \mathbb{R}), O(n))$, $(O(n), O(n-1))$,...

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$$L \curvearrowright G/H, \;\; H \curvearrowright G/L, \;\; \mathrm{diag}(G) \curvearrowright (G\times G)/(H\times L)$$

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• A pioneering work on $L \backslash G/H$ in non-symmetric setting.

• $L \backslash G/H$ in the symmetric case is well-studied by Flensted-Jensen ('78), Hoogenboom ('83), T. Matsuki ('95, '97).

Definition (Generalized Cartan decomposition)

G: *conn. cpt Lie gp,*

T: *maximal torus of* G*,*

L, H: *Levi subgps containing* T*,*

σ: *Chevalley–Weyl involution of* G *w.r.t.* T*.*

($\sigma^2 = \mathrm{id}_G$ *and* $\sigma(t) = t^{-1}\ \forall t \in T$*.)*

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If there exists $B \subset G^\sigma$ s.t.

$$L \times B \times H \to G \text{ is surjective},$$

then we call $G = LBH$ a generalized Cartan decomposition.

Generalized Cartan decomposition

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σ acts on three cpx mfds

$$G/L, \quad G/H, \quad (G \times G)/(L \times H)$$

as anti-holomorphic diffeomorphisms.

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Suppose that we have $G = LBH$ for some $B \subset G^{\sigma}$.

Suppose that $G = LBH$ holds.

Kobayashi's triunity principle

Suppose that $G = LBH$ holds.
Then we obtain three strongly visible actions.

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Furthermore, we can obtain three M.F. theorems by the propagation theorem.

$$\mathrm{Ind}_L^G \chi_L \downarrow_H, \quad \mathrm{Ind}_H^G \chi_H \downarrow_L, \quad \mathrm{Ind}_L^G \chi_L \otimes \mathrm{Ind}_H^G \chi_H.$$

Here χ_L and χ_H are unitary characters of L and H.

Theorem (-T ('12))

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Remark

In the type A case ($G = U(n)$), the theorem is due to Kobayashi ('07).

By the theorem, we can obtain a classification of visible actions on flag varieties and find that

$$\text{M.F.} \Leftrightarrow \text{visible} \Leftrightarrow \text{spherical}.$$

More precisely:

Corollary (-T ('12))

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Then the following 10 conditions are equivalent.

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- *H acts on G/L strongly visibly.*

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- $\exists B \subset G^\sigma$ s.t. $G = LBH$.
- *H acts on G/L strongly visibly.*
- *L acts on G/H strongly visibly.*
- *diag(G) acts on $(G \times G)/(L \times H)$ strongly visibly.*

Corollary (continued)

- *The restriction of* $\mathrm{Ind}_L^G \chi_L$ *to* H *is M.F.*

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- $\mathrm{Ind}_L^G \chi_L \otimes \mathrm{Ind}_H^G \chi_H$ *is M.F.*

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- $G/H \simeq G_{\mathbb{C}}/P_H$ *is* $L_{\mathbb{C}}$*-spherical.*
- $(G \times G)/(L \times H) \simeq (G_{\mathbb{C}} \times G_{\mathbb{C}})/(P_L \times P_H)$ *is diag*$(G_{\mathbb{C}})$*-spherical.*

Here, spherical $\Leftrightarrow$ *Borel subgp has an open orbit.*

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- *Classification of M.F. tensor product rep'ns in the maximal parabolic setting was given by Littelmann ('94).*
- *Classification of M.F. tensor product rep'ns in the general setting was completed by Stembridge ('03) by a combinatorial method.*

Visible action

- We have a classification of visible actions on generalized flag varieties for any type.
- Regarding reductive group-actions on generalized flag varieties, we have

$$\text{M.F.} \Leftrightarrow \text{visible} \Leftrightarrow \text{spherical}.$$

Suppose that $G = LBH$ holds.
Then we obtain three strongly visible actions.

$$H \curvearrowright G/L,\ L \curvearrowright G/H,\ \mathrm{diag}(G) \curvearrowright (G\times G)/(L\times H).$$

Furthermore, we can obtain three M.F. theorems by the propagation theorem.

$$\mathrm{Ind}_L^G \chi_L \downarrow_H, \quad \mathrm{Ind}_H^G \chi_H \downarrow_L, \quad \mathrm{Ind}_L^G \chi_L \otimes \mathrm{Ind}_H^G \chi_H.$$

Here χ_L and χ_H are unitary characters of L and H.

Theorem (Kobayashi ('13) (written again))

G: *Lie gp,*

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V is M.F. if the following hold.

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"*Propagation theorem of M.F. property*"

"Small" M. F. rep'n
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M.F. rep'n
$\mathcal{O}(X, \mathcal{W})$ $\xleftarrow{\text{Propagation th'm}}$

Visible action
$G \curvearrowright X$

We can reduce complicated M. F. th'ms to a pair of data:

- visible actions on cpx mfds, and
- much simpler M. F. th'ms

Motivation

M.F. rep'n $\mathcal{O}(X, \mathcal{W})$ $\xleftarrow{\text{Propagation th'm}}$

- "Small" M. F. rep'n $\mathcal{W}_x$
- Visible action $G \curvearrowright X$

We can reduce complicated M. F. th'ms to a pair of data:

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- much simpler M. F. th'ms
  (*seeds* of M. F. rep'ns by Kobayashi).

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Problem

G: cpt. Lie gp,
V_1, V_2: irr. rep's of G with $V_1 \otimes V_2$ M. F.
Find visible actions and seeds for M. F. tensor products $V_1 \otimes V_2$.

Theorem (Kobayashi ('07))

$G = U(n)$; L, H: Levi subgps of G.
The table gives a classification of (G, L, H) satisfying one of the equivalent conditions:

- *$L \curvearrowright G/H$ is strongly visible,*
- *$H \curvearrowright G/L$ is strongly visible,*
- diag$(G) \curvearrowright (G \times G)/(H \times L)$ *is strongly visible.*

Table : Visible actions on generalized flag varieties of type A

G	L	H
$U(n)$	$U(n)$ *or* $U(1) \times U(n-1)$	*arbitrary Levi subgp*
	$U(p) \times U(q)$	$U(n_1) \times U(n_2) \times U(n_3)$

$\min\{p, q\} \leq 2$ *or* $\min\{n_1, n_2, n_3\} \leq 1$

$(p + q = n_1 + n_2 + n_3 = n)$.

Classification of M.F. representation

Theorem (Kobayashi ('07))

For M.F. tensor product rep'ns of $U(n)$, a classification of seeds is given as follows.

- *1-dim'l rep'n.*
- $\Lambda^i \mathbb{C}^n \downarrow_{\mathbb{T}^n}$.
- $S^i \mathbb{C}^n \downarrow_{\mathbb{T}^n}$.
- $V_{2\varpi_k} \downarrow_{U(n_1)\times U(n_2)\times U(n_3)}$.

Here ϖ_k $(1 \leq k \leq n-1)$ is a fundamental weight of $U(n)$.

Theorem (-T ('13))

$G = SO(N)$, L, H: Levi subgps of G.
A classification of strongly visible actions $L \curvearrowright G/H$, $H \curvearrowright G/L$, $G \curvearrowright (G \times G)/(L \times H)$ is given as follows.

Table : Visible action on orthogonal generalized flag varieties

G	L	H
$SO(N)$	$SO(N)$	*arbitrary Levi subgp*
	$U(j) \times SO(N-2j)$	$U(1) \times SO(2N-2)$
	$U([N/2])$	$U([N/2])$ *or* $\xi(U([N/2]))$

(*continued on the next page*)

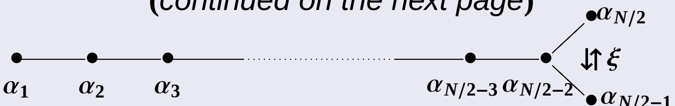

Figure : Dynkin diagram of type $D_{N/2}$

Theorem (continued)

Table : Visible actions on orthogonal generalized flag varieties

G	L	H
$SO(2n)$	$U(1) \times SO(2n-2)$	$U(j) \times U(n-j)$ $\xi(U(j) \times U(n-j))$
	$U(n)$ or $\xi(U(n))$	$U(1) \times U(n-1)$ $\xi(U(1) \times U(n-1))$ $U(1) \times U(1) \times SO(2n-4)$ $U(2) \times SO(2n-4)$ $U(3) \times SO(2n-6)$
	$U(4)$	$\xi(U(2) \times U(2))$
	$\xi(U(4))$	$U(2) \times U(2)$

Classification of M.F. representation

Theorem (-T ('13))

For M.F. tensor product rep'ns of $SO(N)$ (or $Spin(N)$), a classification of seeds is given as follows.

(1) *1-dim'l rep'n.*

(2) $\mathbb{C}^N \downarrow_{T^{[N/2]}}$ *or* $\mathrm{Spin}_N \downarrow_{T^{[n/2]}}$.

(3) $\Lambda^i(\mathbb{C}^N) \downarrow_{U(j)\times SO(N-2j)}$.

(4) $\mathrm{Spin}_N \downarrow_{\{\pm 1, \pm\sqrt{-1}\}\cdot Spin(N-2)}$.

Remark (written again)

- *For the type A case ($G = U(n)$), this corollary is due to Kobayashi ('07).*
- *The equivalence: M.F. $\Leftrightarrow$ spherical was proved by Vinberg–Kimel'fel'd ('78).*
- *Classification of M.F. tensor product rep'ns in the maximal parabolic setting was given by Littelmann ('94).*
- *Classification of M.F. tensor product rep'ns in the general setting was completed by Stembridge ('03) by a combinatorial method.*

A classification of M. F. tensor product $V_{\lambda_1} \otimes V_{\lambda_2}$ for $G = SO(2n+1)$.

(i) $(\lambda_1, \lambda_2) = (s\omega_n, t\omega_n)$ or $(s\omega_1, t\omega_j)$ with $1 \leq j \leq n$ and $s, t \in \mathbb{N}$.

(ii) $\lambda_1 = 0, \omega_1$ or ω_n; λ_2 is arbitrary.

(iii) $\lambda_1 = \omega_i$ or $2\omega_n$; $\lambda_2 = t\omega_j$ with $1 \leq i, j \leq n$ and $t \in \mathbb{N}$.

(iv) $\lambda_1 = \omega_n + s\omega_j$; $\lambda_2 = t\omega_1$ with $1 \leq j \leq n$ and $s, t \in \mathbb{N}$.

A classification of M. F. tensor product $V_{\lambda_1} \otimes V_{\lambda_2}$ for $G = SO(2n)$.

(i') $(\lambda_1, \lambda_2) = (s\omega_1, t\omega_j + u\omega_{n-1})$ or $(s\omega_1, t\omega_j + u\omega_n)$ with $1 \leq j \leq n$ and $s, t, u \in \mathbb{N}$,
$\lambda_1 = s\omega_{n-1}$ or $s\omega_n$; $\lambda_2 = t\omega_3$, $t\omega_1 + u\omega_2$, $t\omega_1 + u\omega_{n-1}$, $t\omega_1 + u\omega_n$ or $t\omega_{n-1} + u\omega_n$ with $s, t, u \in \mathbb{N}$, or
$\lambda_1 = s\omega_{5-\epsilon}$; $\lambda_2 = t\omega_2 + u\omega_{2+\epsilon}$ with $n = 4$ and $\epsilon = 1$ or 2.

(ii') $\lambda_1 = 0$, ω_1, ω_{n-1} or ω_n; λ_2 is arbitrary.

(iii') $\lambda_1 = \kappa\omega_i$; $\lambda_2 = t\omega_j$, where $t \in \mathbb{N}$ and κ, i, j satisfy one of the following three conditions.

- (iii'-1) $\kappa = 1$ and $i + j \leq n$.
- (iii'-2) $\kappa = 1$, $1 \leq i \leq n$ and $j = n - 1$ or n.
- (iii'-3) $\kappa = 2$, $i = n - 1$ or n and $1 \leq j \leq n$.

Classification of M.F. representation

Theorem (-T ('13))

For M.F. tensor product rep'ns of $SO(N)$ (or $Spin(N)$), a classification of seeds is given as follows.

(1) *1-dim'l rep'n.*

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(3) $\Lambda^i(\mathbb{C}^N) \downarrow_{U(j)\times SO(N-2j)}$.

(4) $\mathrm{Spin}_N \downarrow_{\{\pm 1, \pm\sqrt{-1}\}\cdot Spin(N-2)}$.

Summary

Visible action

- We have a classification of visible actions on generalized flag varieties for any type.
- Regarding reductive group-actions on generalized flag varieties, we have

$$\text{M.F.} \Leftrightarrow \text{visible} \Leftrightarrow \text{spherical}$$

Seed

- We have a classification of seeds for M.F. tensor product rep'ns of $U(n)$ and $SO(n)$.

Thank you very much for your kind attention!

Shukran

Spasiba

Blagodarya

Thank you

Dziekuje

Kheili mamnun

Merci

Dzjakuj

Dankie

Arigatou